Integrable particle systems and Macdonald processes

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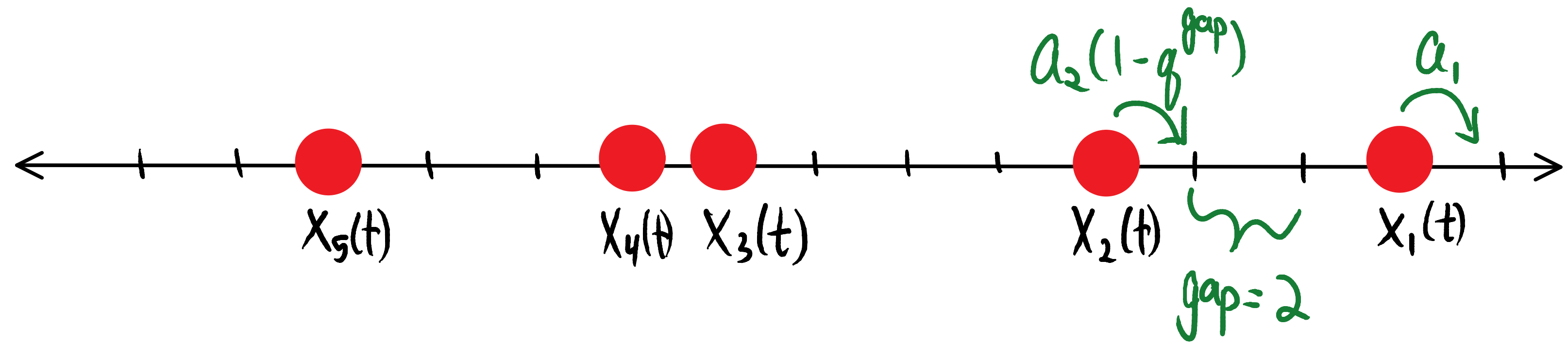
Lecture 4

- Expectations of q-TASEP observables solve integrable many body systems which can be solved via variant of Bethe ansatz
- Limit to directed polymers shows this is rigorous replica method
- Also applies to discrete q-TASEPs, q-PushASEP, and ASEP

<u>q-TASEP</u>:



$q \in (0,1)$

$a_i \in (0,\infty)$

Restrict to N particle state space

$$\mathbb{X}^N = \left\{ \vec{x} := (x_0, x_1, \dots, x_N) \in \{\infty\} \times \mathbb{Z}^N : \infty = x_0 > x_1 > x_2 > \dots > x_N \right\}$$

Generator acts on $f : \mathbb{X}^N \to \mathbb{R}$ as

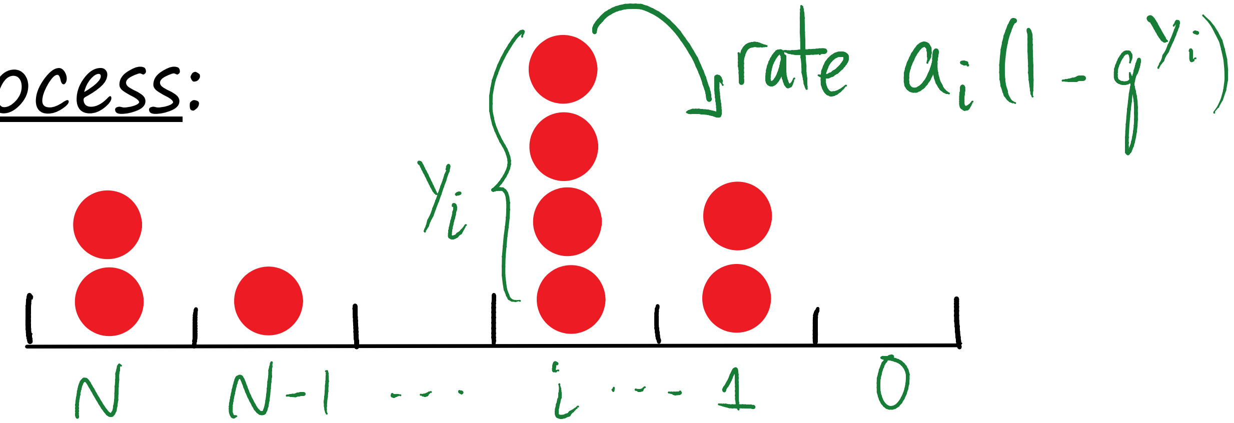
$$= (x_0, x_1, \dots, x_i + 1, \dots, x_N)$$

$$\left(L^{q\text{-TASEP}} f\right)(\vec{x}) = \sum_{i=1}^{N} a_i \left(1 - q^{x_{i-1} - x_i - 1}\right)\left(f(\vec{x}_i^{+}) - f(\vec{x}) \right)$$

Natural initial condition is <u>step</u> where $x_i(0) = -i \quad , i \geq 1$

(When q=0, we recover the usual TASEP)

q-Boson particle process:



N+1 site state space

$$\mathbb{Y}^N = \{\vec{y} = (y_0, y_1, \dots, y_N) \in \mathbb{Z}_{\geq 0}^{\{0,1,\dots,N\}}\}$$

$$\mathbb{Y}^N_k = \{\vec{y} \in \mathbb{Y}^N : \sum y_i = k\}$$

Generator acts on $h : \mathbb{Y}^N \to \mathbb{R}$ as

$$= (y_0, \dots, y_{i-1}+1, y_i - 1, \dots, y_N)$$

$$(L^{q\text{-TAZRP}} h)(\vec{y}) = \sum_{i=1}^{N} a_i (1 - q^{y_i}) \left(h(\vec{y}^{\,i,i-1}) - h(\vec{y}) \right)$$

[Sasamoto-Wadati '98] stochastic representation of q-Bosons

[Balazs-Komjathy-Seppalainen '08] stationary 1/3 exponent

Duality: Suppose $X(t) \in \mathbb{X}$ and $y(t) \in \mathbb{Y}$ independent Markov processes and $H : \mathbb{X} \times \mathbb{Y} \to \mathbb{R}$. Then $X(t)$ and $y(t)$ are dual with respect to H if for all x, y, and t

$$\mathbb{E}^{x}\left[H(x(t), y)\right] = \mathbb{E}^{y}\left[H(x, y(t))\right].$$

- Duality leads to hidden evolution equations for expectations of observables corresponding to the duality function.

Theorem [Borodin-C-Sasamoto '12]: q-TASEP $\vec{X}(t) \in \mathbb{X}^N$ and q-Bosons $\vec{y}(t) \in \mathbb{Y}^N$ are dual with respect to

$$H(\vec{x},\vec{y}) = \prod_{i=0}^{N} q^{(x_i+i)y_i}$$

(convention that if $y_0 > 0$, $H \equiv 0$)

Proof: Suffices to show that

$$L^{q\text{-TASEP}} H(\vec{x},\vec{y}) \overset{?}{=} L^{q\text{-TAZRP}} H(\vec{x},\vec{y})$$

$$\sum_{i=1}^{N} a_i \left(1 - q^{x_{i-1}-x_i-1}\right)\left(q^{y_i}-1\right)\prod_{j=0}^{N} q^{(x_j+j)y_j} = \sum_{i=1}^{N} a_i \left(1-q^{y_i}\right)\left(q^{x_{i-1}-x_i-1}-1\right)\prod_{j=0}^{N} q^{(x_j+j)y_j}$$

$\square$

Purpose of duality (for us):

If $\vec{y} = (0,0,\ldots,0,k)$ then

$$h(t;\vec{y}) := \mathbb{E}^{\vec{x}}\left[H(\vec{x}(t),y)\right] = \mathbb{E}^{\vec{x}}\left[q^{k(x_N(t)+N)}\right]$$

Duality implies that for $\vec{x}$ fixed, $h(t;\vec{y})$ solves the

<u>True evolution equation</u>:

$$\begin{cases} \frac{d}{dt} h(t;\vec{y}) = L^{q\text{-TAZRP}} h(t;\vec{y}) \\ h(0;\vec{y}) = H(\vec{x},\vec{y}) \quad \left[= h_0(\vec{y})\right] \end{cases}$$

True evolution equation splits according to number of particles

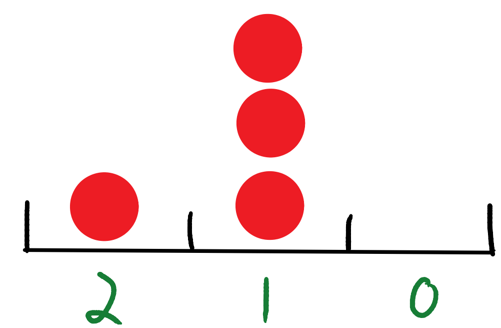
$$W_{\geq 0}^K := \left\{ \vec{n} = (n_1, \ldots, n_K) \in \mathbb{Z}_{\geq 0}^K : \; n_1 \geq n_2 \geq \cdots \geq n_K \geq 0 \right\}$$

Encode $\vec{y} \in Y_K^N$ by an ordered list of particle locations

$$\begin{array}{ccccc} & \vec{y} & \longmapsto & \vec{n}(\vec{y}) & \\ Y_K^N \ni & & & & \in W_{\geq 0}^K \\ & \vec{y}(\vec{n}) & \longleftarrow & \vec{n} & \end{array}$$

Example: $N = 2$, $K = 4$

$$\vec{y} = (0, 3, 1) \longleftrightarrow \vec{n} = (1, 1, 1, 2)$$



We can encode true evolution equation in the $\vec{n}$ coordinates by writing $g(t;\vec{n}) := h(t;\vec{y}(\vec{n}))$, $g_0(\vec{n}) := h_0(\vec{y}(0))$

- $k=1$: single particle, so $\vec{n} = (n)$, then

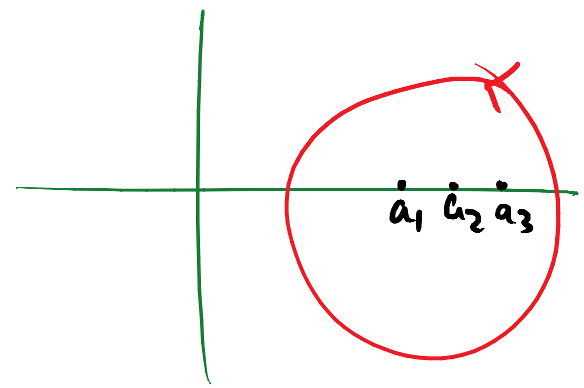
$$\begin{cases} \frac{d}{dt} g(t;n) = a_n(1-q)\nabla g(t;n) \\ g(t;0) \equiv 0 \\ g(0;n) = g_0(n) \end{cases}$$

$$\left[(\nabla f)(n) := f(n-1) - f(n) \right]$$

For step initial data $X_i + i = 0$ so $H(\vec{x},\vec{y}) \equiv 1$ and so too $g_0 \equiv 1$

Claim: $\mathbb{E}^{\text{step}}\left[q^{X_n(t)+n} \right] = g(t;n) = \frac{-1}{2\pi i} \oint g_z(t;n) \frac{dz}{z}$

where $g_z(t;n) = \prod_{m=1}^{n} \frac{a_m}{a_m - z} e^{(q-1)tz}$



Proof: Check free equation, zero boundary condition, and initial data. $\square$

- $k=2$: two particles, so $\vec{n} = (n_1 \geq n_2)$
  - If $n_1 > n_2$

acts as ∇ on n_i coordinate

$$\frac{d}{dt} g(t;\vec{n}) = \sum_{i=1}^{2} a_{n_i}(1-q)\nabla_i\, g(t;\vec{n})$$

  - If $n_1 = n_2$

$$\frac{d}{dt} g(t;\vec{n}) = a_{n_2}(1-q^2)\nabla_2\, g(t;\vec{n})$$

Not constant coefficient, so unclear how to solve...

- $k>2$: there are different equations for each type of clustering (i.e., many body interactions)

Proposition: (Free evolution eqn with k-1 boundary conditions):

If $u : \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}^{k} \longrightarrow \mathbb{R}$ solves

- For all $\vec{n} \in \mathbb{Z}_{\geq 0}^{k}$, $t \geq 0$,

Free evolution eqn $$\frac{d}{dt} u(t;\vec{n}) = \sum_{i=1}^{k} a_{n_i} (1-q) \nabla_i \, u(t;\vec{n})$$

- For all $\vec{n} \in \mathbb{Z}_{\geq 0}^{k}$ such that $n_i = n_{i+1}$

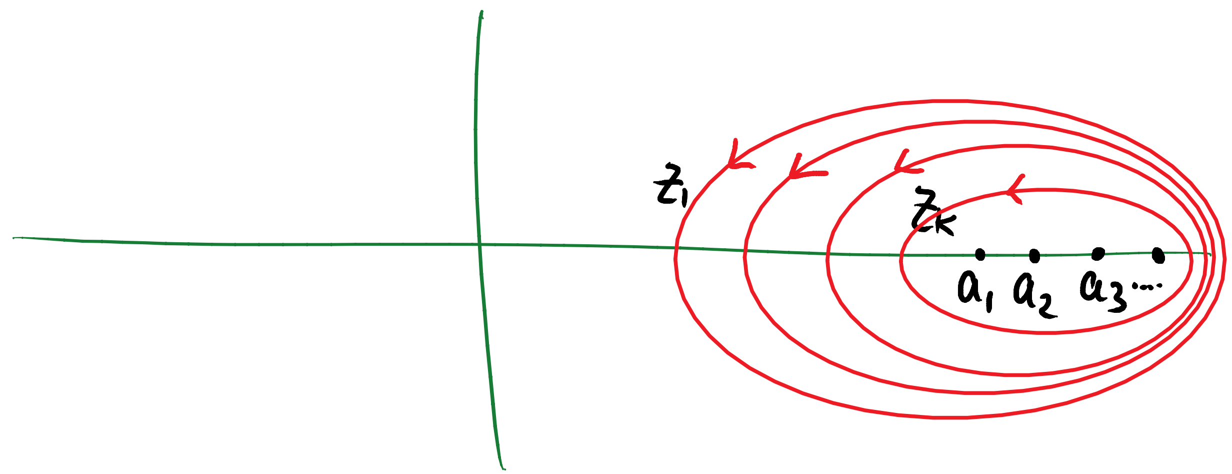
Boundary conditions $$(\nabla_i - q \nabla_{i+1}) \, u(t;\vec{n}) = 0$$

- For all $\vec{n} \in \mathbb{Z}_{\geq 0}^{k}$ such that $n_k = 0$, $u(t;\vec{n}) \equiv 0$
- For all $\vec{n} \in W_{\geq 0}^{k}$, $u(0;\vec{n}) = g_0(\vec{n})$

Then, restricted to $\vec{n} \in W_{\geq 0}^{k}$, $g(t;\vec{n}) = u(t;\vec{n})$.

<u>Theorem</u>: For step initial condition (i.e., $g_0(\vec{n}) \equiv 1$) we have

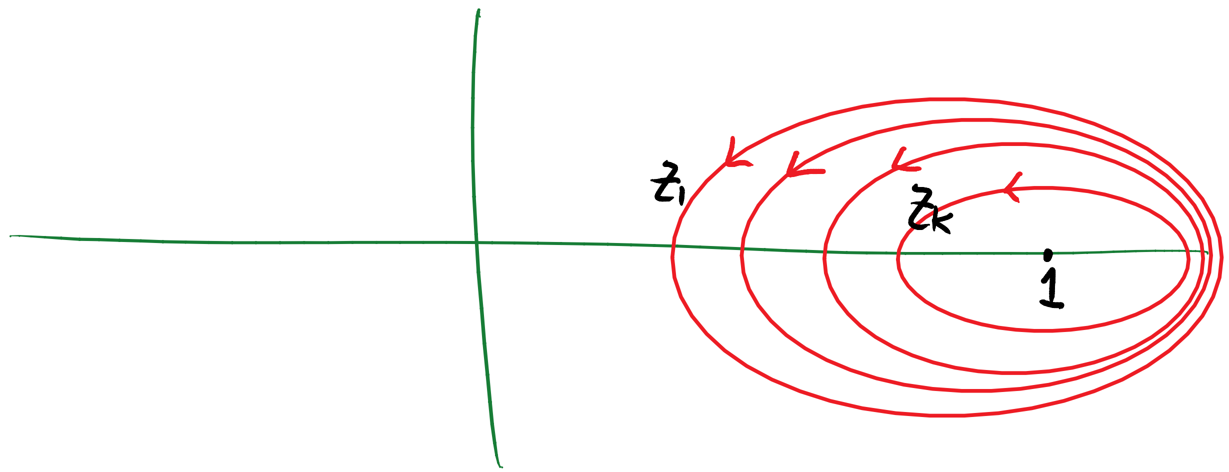
$$\mathcal{U}(t;\vec{n}) = \frac{(-1)^k q^{k(k-1)/2}}{(2\pi i)^k} \oint \cdots \oint \prod_{1 \le A < B \le k} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} g_{z_j}(t;n_j) \frac{dz_j}{z_j}$$



Proof: Only new aspect is boundary condition. Applied to integrand brings out factor of $z_i - q z_{i+1}$. Contour symmetry and integrand asymmetry shows integral is zero. $\square$

Implies joint moment formulas. For example, if all $a_i \equiv 1$

$$\mathbb{E}^{\text{step}}\left[q^{k(x_n(t)+n)} \right] = \frac{(-1)^k q^{k(k-1)/2}}{(2\pi i)^k} \oint \cdots \oint \prod_{1 \le A < B \le k} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} \frac{e^{(q-1)t z_j}}{(1-z_j)^n} \frac{dz_j}{z_j}$$



Success in using moments to asymptotically study one-point distribution, though multi-point distributions remain open

True evolution equation also equivalent to a certain

<u>q-deformed discrete delta Bose gas</u>

$$\frac{d}{dt} g(t;\vec{n}) = H g(t;\vec{n})$$

with Hamiltonian

$$H = (1-q)\left[\sum_{j=1}^{k} \nabla_j + (1-q^{-1}) \sum_{1\le i<j\le k} \delta_{n_i=n_j}\, q^{j-i} \nabla_j \right]$$

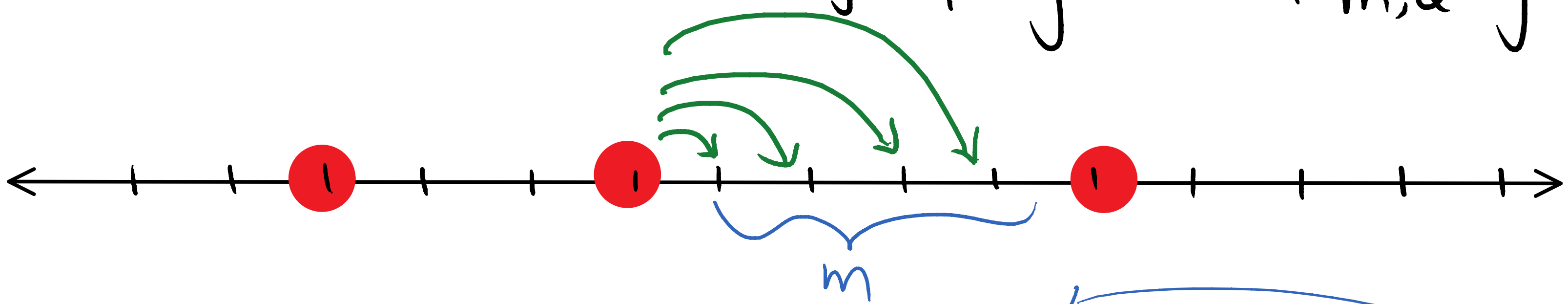
subject to Bosonic symmetry and zero boundary condition

Integrability (equiv. to free eqn with k−1 B.C.s) not obvious

for this system (Note: not all delta Bose gases are integrable)

(Parallel) Geometric discrete time q-TASEP [Borodin-C '13]:

$$\mathbb{P}(\text{jump}=j) = P_{m,\alpha}(j)$$



$q \in (0,1)$, $\alpha \in (0,1)$

$$P_{m,\alpha}(j) = \alpha^j (\alpha;q)_{m-j} \frac{(q;q)_m}{(q;q)_{m-j}(q;q)_j} \mathbb{1}_{0 \le j \le m}$$

$$\left[(a;q)_n = \prod_{j=0}^{n-1} (1 - a q^j) \right]$$

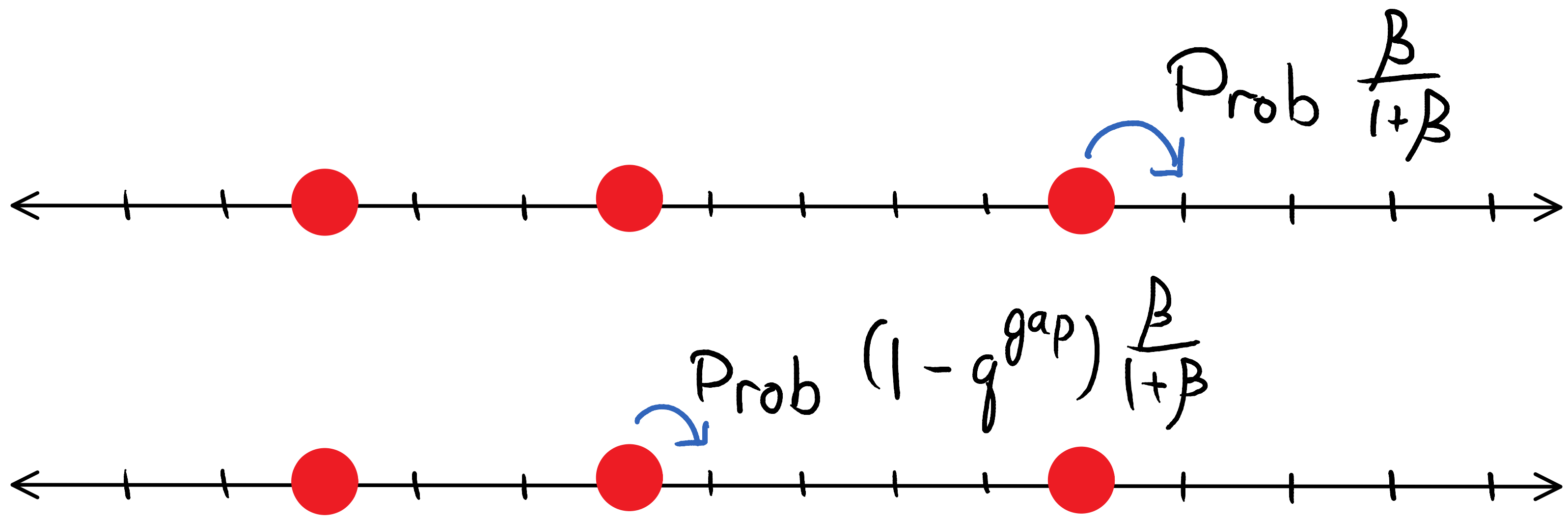
At q=0 -> parallel geometric TASEP with blocking

[Warren-Windridge '09]

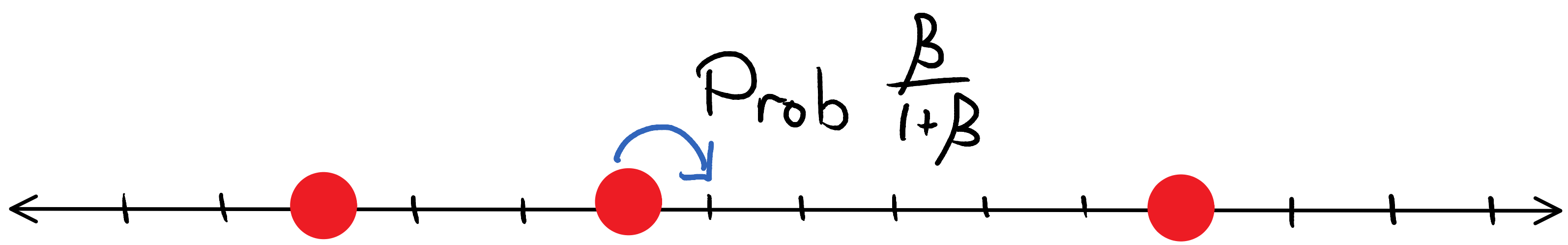
(Sequential) Bernoulli discrete time q-TASEP [Borodin-C '13]:

$q \in (0,1)$, $\beta \in (0,\infty)$

1)



2)



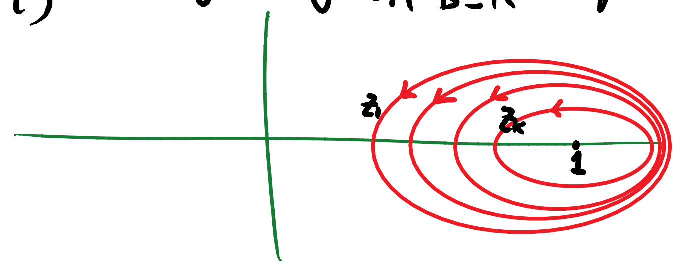
3) ...

At $q=0$ -> sequential Bernoulli TASEP [Borodin-Ferrari '08]

q-TASEP joint moments satisfy various many body systems

<u>Theorem</u> [Borodin-C '13]: For $n_1 \geq n_2 \geq \cdots \geq n_k > 0$

$$\mathbb{E}^{\text{step}}\left[\prod_{j=1}^{k} q^{x_{n_j}(t)+n_j}\right] = \frac{(-1)^k q^{k(k-1)/2}}{(2\pi i)^k} \oint \cdots \oint \prod_{1\leq A<B\leq k} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} \frac{1}{(1-z_j)^{n_j}} \frac{f(qz_j)}{f(z_j)} \frac{dz_j}{z_j}$$



$$f(z) = \begin{cases} e^{tz}, & \text{Poissonian continuous } q\text{-TASEP} \\ \left(\frac{1}{(\alpha z; q)_\infty}\right)^t, & \text{Geometric discrete } q\text{-TASEP} \\ (1+\beta z)^t, & \text{Bernoulli discrete } q\text{-TASEP} \end{cases}$$

q-TASEP satisfies:

$$\begin{cases} d\,q^{x_n(t)+n} = (1-q)\nabla q^{x_n(t)+n}\,dt + q^{x_n(t)+n}\,dM_n(t) \\ q^{x_n(0)+n} \equiv 1 \ \text{(step)}, \quad q^{x_0(t)+0} \equiv 0 \ (x_0 = \infty) \end{cases}$$

Martingale

<u>Theorem [Borodin-C '11]</u>: For q-TASEP with step init. cond.

scale $q = e^{-\varepsilon}$, $t = \varepsilon^{-2}\tau$, $x_n(t) = \varepsilon^{-2}\tau - (n-1)\varepsilon^{-1}\log\varepsilon^{-1} - \varepsilon^{-1}F_\varepsilon(\tau,n)$

and call $Z_\varepsilon(\tau,n) = \exp\left\{-\frac{3\tau}{2} + F_\varepsilon(\tau,n)\right\}$. Then as $\varepsilon \searrow 0$,

$Z_\varepsilon(\cdot,\cdot) \Rightarrow Z(\cdot,\cdot)$ where Z solves the <u>semi-discrete SHE</u>:

$$\begin{cases} dZ(\tau,n) = \nabla Z(\tau,n)\,d\tau + Z(\tau,n)\,dB_n(\tau) \\ Z(0,n) = \mathbb{1}_{n=0}, \quad Z(\tau,0) \equiv 0 \end{cases}$$

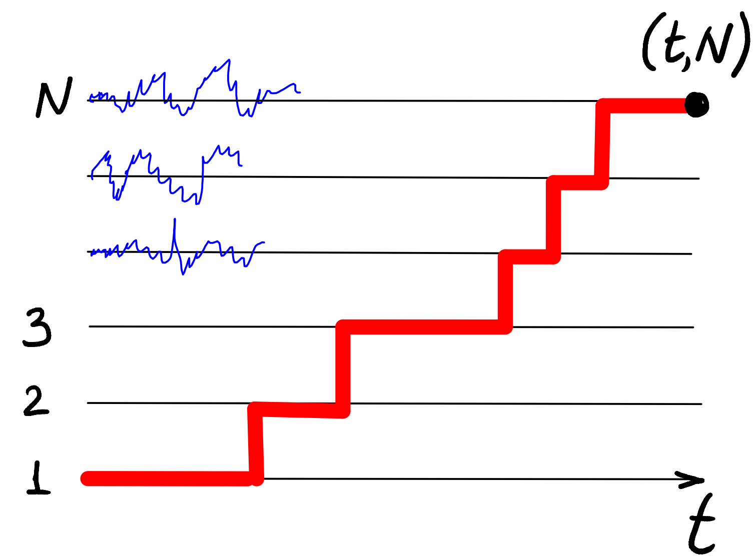
ind. BM's

Partition function for a semi-discrete directed random polymer

$$Z_t^N = \int_{0<s_1<\dots<s_{N-1}<t} e^{B_1(0,s_1)+B_2(s_1,s_2)+\dots+B_N(s_{N-1},t)}\, ds_1 \dots ds_{N-1}$$

$B_1, \dots, B_N$ are independent Brownian motions

$$B_k(\alpha,\beta) := B_k(\beta) - B_k(\alpha) = \int_\alpha^\beta \dot{B}_k(x)\,dx$$



[O'Connell-Yor 2001]

$$u(t,N) := e^{-3t/2} Z_t^N = e^{-3t/2} \int_{0<s_1<\dots<s_{N-1}<t} e^{B_1(0,s_1)+\dots+B_N(s_{N-1},t)} \, ds$$

satisfies

$$\frac{\partial u(t,N)}{\partial t} = \big(u(t,N-1) - u(t,N)\big) + \dot{B}_N(t) \cdot u(N,t)$$

with $u(0,N) = \delta_{1N}$.

This is a discrete analog of the <u>stochastic heat equation</u>

$$u_t = \frac{1}{2} \Delta u + \dot{W} \cdot u$$

where $\dot{W}$ is the space-time white noise.

The path integral is the <u>Feynman-Kac solution</u>

The semi-discrete Brownian directed polymer is exactly solvable.

<u>Theorem (Borodin-Corwin, 2011)</u> The Laplace transform of the polymer partition function Z_t^N can be written as a Fredholm determinant

$$\left\langle e^{-u Z_t^N} \right\rangle = \det\left(\mathbb{1} + K_u\right)_{L^2(\bigcirc)}$$

where

$$K_u(v,v') = \frac{i}{2} \int_{-i\infty+\frac{1}{2}}^{i\infty+\frac{1}{2}} \left(\frac{\Gamma(v-1)}{\Gamma(s+v-1)} \right)^N \frac{u^s e^{vts + \frac{ts^2}{2}}}{s+v-v'} \frac{ds}{\sin \pi s}.$$

<u>Corollary (B-C, B-C-Ferrari, 2011-12)</u> Set $F_t^N = \log Z_t^N$. For any $\varkappa > 0$

$$\lim_{N\to\infty} \mathbb{P}\left\{ \frac{F_{\varkappa N}^N - N\bar{f}_\varkappa}{N^{1/3}} \le r \right\} = F_{GUE}\left(\left(\frac{\bar{g}_\varkappa}{2}\right)^{-1/3} r \right)$$

This leads to a rigorous derivation of $\mathbb{E}\left[e^{-\zeta Z(\tau,n)}\right] = \det(I + \tilde{K}_\zeta)$ and proof that logarithm of semi-discrete SHE has GUE Tracy-Widom scaling limit under $\tau^{1/3}$ scaling (Ferrari's talk)

Under weak noise scaling [Alberts-Khanin-Quastel '12] the semi-discrete SHE converges weakly to the continuum SHE [Moreno Flores-Remenik-Quastel '13]:

$$\partial_t Z(t,x) = \tfrac{1}{2}\partial_x^2 Z(t,x) + Z(t,x)\xi(t,x), \qquad Z(0,x) = \delta_{x=0}$$

(space time white noise)

Thus a second proof of SHE Laplace transform Fredholm det.

[Sasamoto Spohn '10, Amir-C-Quastel '10, Calabrese-Le Doussal-Rosso '10, Dotsenko '10]

Feynman-Kac representation leads to semi-discrete polymer [O'Connell-Yor '01] and continuum random polymer.

Replica method [Molchanov '86, Kardar '87] shows that joint moments $\mathbb{E}\left[\prod_{j=1}^{k} Z(\tau, n_j)\right]$, $\mathbb{E}\left[\prod_{j=1}^{k} Z(t, x_i)\right]$ satisfy delta Bose gases

$$H = \sum_{j=1}^{k} \nabla_j + \sum_{1\le i<j\le k} \mathbb{1}_{n_i=n_j} \quad , \quad H = \frac{1}{2}\sum_{j=1}^{k} \partial_{x_j}^2 + \sum_{1\le i<j\le k} \delta_{x_i=x_j}$$

Both can we written as free evolution eqn. with k-1 B.C.'s and solved by limits of the q-TASEP nested contour formulas.

However, these moments grow like e^{ck^2}, e^{ck^3} and hence do not characterize the distribution of Z (replica trick).

General coupling const. version ($c<0$ repulsive, $c>0$ attractive)

$$\begin{cases} \partial_t u(t;\vec{x}) = \sum_{j=1}^{k} \partial_{x_j}^2 u(t;\vec{x}), \\ (\partial_{x_i} - \partial_{x_{i+1}} - c) u(t;\vec{x}) \Big|_{x_i \nearrow x_{i+1}} = 0, \quad u(0;\vec{x}) = \delta_{\vec{x}=0} \end{cases}$$

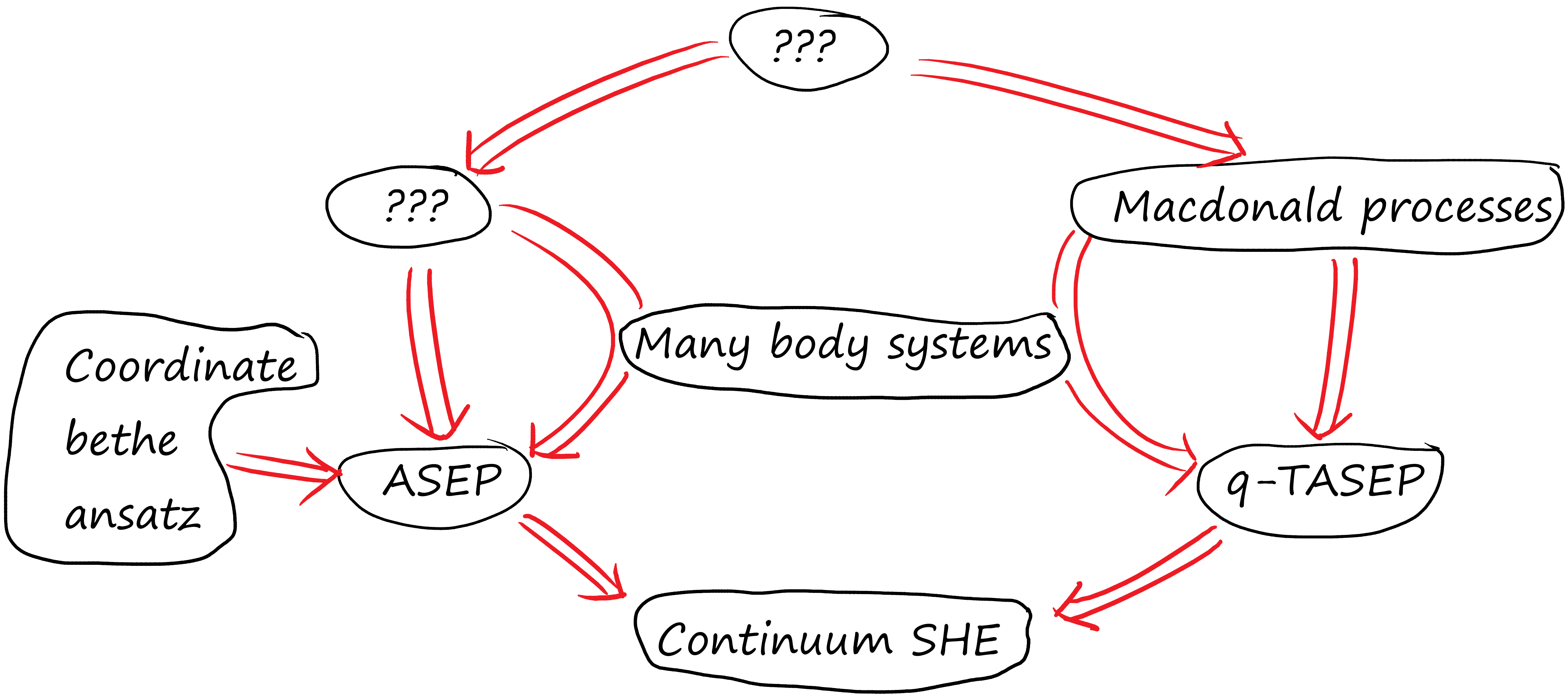
is solved (in $x_1 < x_2 < \cdots < x_k$) by the nested contour integral formula

$$u(t;\vec{x}) = \frac{1}{(2\pi i)^k} \int \cdots \int \prod_{1 \le A < B \le k} \frac{z_A - z_B}{z_A - z_B - c} \prod_{j=1}^{k} e^{x_j z_j + \frac{t}{2} z_j^2} dz_j$$

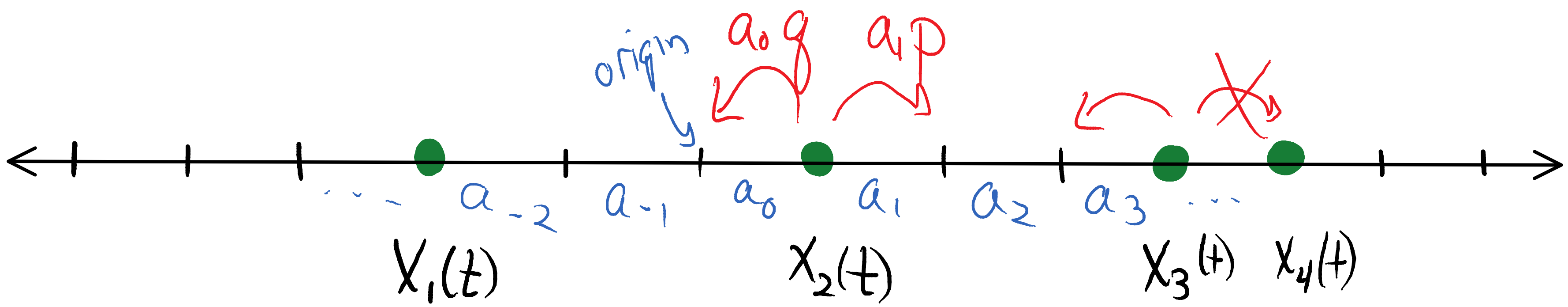
where z_j is integrated over $\alpha_j + i\mathbb{R}$, with $\alpha_1 > \alpha_2 + c > \alpha_3 + 2c > \cdots$

Many body systems approach reveals parallel formulas.

Is there a higher structure which accounts for this?



Asymmetric simple exclusion (particle) process



Particles attempt continuous time random walks, jumping left over bond $i \leftrightarrow i+1$ at rate $a_i q$ and right at rate $a_i p$. If the destination is occupied, the jump is suppressed.

State space for k particles: $W^k = \{ x_1 < x_2 < \cdots < x_k \} \subseteq \mathbb{Z}^k$.

Generator $(L^{k,\mathrm{part}} f)(\vec{x})$ for $\vec{x} \in W^k$.

e.g. k=1 $\quad (L^{1,\mathrm{part}} f)(x) = a_x p [f(x+1) - f(x)] + a_{x-1} q [f(x-1) - f(x)]$

Asymmetric simple exclusion (occupation) process

$\eta(t) = \{\eta_x(t)\}_{x\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$, $\eta_x(t) = \begin{cases} 1 & \text{particle at } x, \text{time } t \\ 0 & \text{otherwise} \end{cases}$

Dynamics: for each y $\begin{cases} \eta \mapsto \eta^{y,y+1} & \text{at rate } a_y p \text{ if } (\eta_y, \eta_{y+1}) = (1,0) \\ \eta \mapsto \eta^{y,y+1} & \text{at rate } a_y q \text{ if } (\eta_y, \eta_{y+1}) = (0,1) \end{cases}$

$$(L^{occ} f)(\eta) = \sum_{y\in\mathbb{Z}} a_y \left(p\,\eta_y(1-\eta_{y+1}) + q(1-\eta_y)\eta_{y+1}\right)\left[f(\eta^{y,y+1}) - f(\eta)\right]$$

Assume that $q \geq p \geq 0$ so $p/q =: \tau \leq 1$ $(p+q=1)$ and $c < a_x < c^{-1}$

<u>Define</u>: $N_x = N_x(\eta) = \sum_{y\leq x} \eta_y$

Theorem [Borodin-C-Sasamoto '12]: For any k>0, the ASEP particle process $\vec{x}(t)$ (with p<->q switched) and the ASEP occupation process $\eta(t)$ are dual with respect to

$$H(\eta,\vec{x}) = \prod_{i=1}^{k} \tau^{N_{x_i-1}(\eta)} \eta_{x_i}$$

$$\left(\text{i.e. } \mathbb{E}^{\eta}\big(H(\eta(t),\vec{x})\big) = \mathbb{E}^{\vec{x}}\big(H(\eta,\vec{x}(t))\big) \text{ for all } \eta\in\{0,1\}^{\mathbb{Z}},\ \vec{x}\in W^k,\ t\geq 0\right)$$

If all bond jump rates parameters $a_i \equiv 1$ then the processes are also dual with respect to

$$G(\eta,\vec{x}) = \prod_{i=1}^{k} \tau^{N_{x_i}(\eta)}$$

Remarks on the duality.

- When p=q, the H-duality describes correlation functions and is much more general.
- When all $a_i \equiv 1$, H-duality shown previously [Schutz '97] via related quantum spin chain $\mathcal{U}_q(sl_2)$- symmetry.
- When k=1, the G-duality is Gartner's microscopic ASEP Hopf-Cole transform.

Proof: Directly from studying the effect of applying the Markov generators to the duality function.

From duality to determinants:

1. Duality lead to system of ODEs for $h(t,\vec{x}) := \mathbb{E}^{\mathfrak{z}}\left[\prod_{i=1}^{k} \tau^{N_{x_i-1}(\mathfrak{z}(t))} \mathfrak{z}_x(t)\right]$

2. For $a_i \equiv 1$ / step initial data, solve ODEs via a "nested contour integral ansatz" (relies on integrability)

3. Combine integral solutions to yield formula for $\mathbb{E}\left[\tau^{n N_x(\mathfrak{z}(t))}\right]$

4. Deform nested-contours to coincide and track residues

5. Form generating function (τ-Laplace transform) and identify Fredhold determinant (Mellin Barnes/Cauchy type).

$$\mathbb{E}\left[\frac{1}{(\zeta \tau^{N_x(t)};\tau)_\infty}\right] = \det(I + K_\zeta)$$

Let's focus on steps 1 and 2.

Duality provides a non-trivial coupled system of ODEs:

Since $\mathbb{E}^{\eta}\left[H(\eta(t),\vec{x})\right] = \mathbb{E}^{\vec{x}}\left[H(\eta,\vec{x}(t))\right] =: h(t,\vec{x})$

$$\frac{d}{dt} h(t,\vec{x}) = \left(L^{k,\mathrm{part}}\right)^{*} h(t,\vec{x}) \quad , \quad h(0,\vec{x}) = H(\eta,\vec{x}).$$

But how to solve? For k>1 the generator depends on $\vec{x}$!

First idea (from Bethe, cf. Tracy-Widom ASEP papers):

Try to solve "free" system of ODEs on all of $\mathbb{Z}^k$ with boundary conditions on W^k.

Proposition: If $u : \mathbb{Z}^k \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ solves

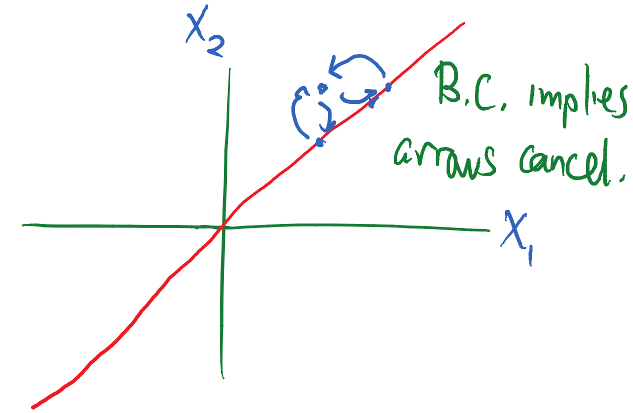
'Free' ODEs

1. $$\frac{d}{dt} u(t, \vec{x}) = \sum_{i=1}^{k} (L^{1,\text{part}})_i^* \, u(t, \vec{x})$$

Boundary condition

2. For all $\vec{x} \in \mathbb{Z}^k$: $x_{i+1} = x_i + 1$ for some i,

$$p\, u(t, \vec{x}_{i+1}^{-}) + q\, u(t, \vec{x}_i^{+}) = u(t, \vec{x})$$



Initial data

3. For all $\vec{x} \in W^k$, $u(0, \vec{x}) = H(\eta, \vec{x})$

Then for all $t \geq 0$, $\vec{x} \in W^k$, $h(t, \vec{x}) = u(t, \vec{x})$

(Note: Since system of ODEs is infinite, we must also impose an exponential growth condition; and we can weaken initial data to weakly converge, as is useful in our contour integral formulas we find)

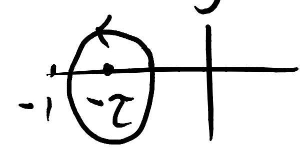
Assume from now on step initial condition ($\eta_x = \mathbb{1}_{x\geq 1}$) and $a_i \equiv 1$

How to solve this system of ODEs?

$$K=1: \quad h_z(t,x) := \exp\left\{-\frac{z(p-q)^2}{(1+z)(p+qz)}t\right\}\left(\frac{1+z}{1+z/\tau}\right)^{x-1}\frac{1}{\tau+z}$$

solves the "free" evolution eqn. for all $z \in \mathbb{C}/\{-\tau\}$.

$$u(t,x) = \mathbb{E}^{\text{step}}\left[\tau^{N_{x-1}(\eta(t))}\eta_x(t)\right] = \frac{1}{2\pi i}\oint h_z(t,x)\,dz$$

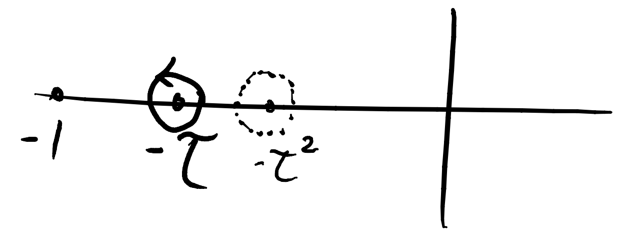


Proof: Check by residues that $u(0,x) = \tau^{x-1}\mathbb{1}_{x\geq 1}$

For $k>1$ we use an idea inspired from the theory of Macdonald processes -> "nested contour integral ansatz"

<u>Theorem</u> [Borodin-C-Sasamoto '12]: For all $k \geq 1$,

$$U(\vec{x};t) = \frac{\tau^{k(k-1)/2}}{(2\pi i)^k} \int \cdots \int \prod_{1\leq A<B\leq k} \frac{z_A - z_B}{z_A - \tau z_B} \prod_{i=1}^{k} h_{z_i}(t,x_i)\, dz_i$$

Where contour of integration is 

to avoid poles of $z_A - \tau z_B$.

Restricting to $\vec{x} \in W^k$ yields:

$$\mathbb{E}^{step}\left[\prod_{i=1}^{k} \tau^{N_{x_i-1}(\eta(t))} \eta_x(t)\right]$$

Assume $x_2 = x_1 + 1$ and check boundary cond.
Try to apply it to integrand $\Rightarrow$ brings out factor $(z_1 - \tau z_2)$. Cancels with $\prod_{A<B}$ term.
What remains is $\iint (z_1 - z_2) G(z_1) G(z_2) = 0$.

Suitable combinations of $\mathbb{E}^{step}\left[\prod_{i=1}^{k} \tau^{N_{x_i-1}(\eta(t))}\eta_x(t)\right]$ yields $\mathbb{E}^{step}\left[\tau^{n N_x(\eta(t))}\right]$

<u>Theorem</u> [Borodin-C-Sasamoto '12]: For step initial condition ASEP with $a_i \equiv 1$ and $p<q$ (hence $\tau = p/q < 1$, $\gamma = q-p > 0$)

$$\mathbb{E}\left[\frac{1}{(\zeta\tau^{N_x(\eta(t))};\tau)_\infty}\right] = \det(I+K_\zeta)_{L^2(\cdot)}$$

Mellin Barnes type

$$= \frac{1}{(\zeta;\tau)_\infty}\det(I-\zeta\tilde{K})_{L^2(\cdot)}$$

Cauchy type

$$K_\zeta(w,w') = \frac{1}{2\pi i}\int \frac{\pi\, ds}{\sin(-\pi s)}(-\zeta)^s \frac{g(w)}{g(\tau^s w)}\frac{1}{w'-\tau^s w}$$

$$g(w) = e^{\gamma t \frac{\tau}{\tau+w}}\left(\frac{\tau}{\tau+w}\right)^x$$

$$\tilde{K}(w,w') = \frac{e^{\varepsilon'(w)t}}{\tau w - w'}$$

$$\varepsilon'(w) = -\frac{1}{q}\frac{w\gamma^2}{(1+w)(\tau+w)}$$

Corollary [Tracy-Widom '09, Borodin-C-Sasamoto '12]:

$$\lim_{t\to\infty} \mathbb{P}^{step}\left(\frac{N_0(\eta(t/\gamma)) - t/4}{t^{1/3}} \geq -r \right) = F_{GUE}(2^{4/3} r)$$

Recovering the celebrate Tracy-Widom / Johansson result.

Remarks:

- Mellin Barnes Fredhold det. new and easy for asymptotics
- Inversion of Cauchy Fredholm det. equivalent to initial det. in [Tracy-Widom '09]
- Completely parallel to q-TASEP formulas

Coordinate approach of [Tracy-Widom '08-'09]:

- Study k particle ASEP and use coordinate Bethe ansatz (cf. [Schutz '97] for k=2) to compute Green's functions.
- Manipulate formulas to extra one-point marginal.
- Approach step initial condition by taking k to infinity and observe an integral transform of Cauchy type Fredholm det.
- Functional analysis to rework for asymptotic analysis.

Using k-particle Green's functions can write solution of duality ODEs as k! k-fold contour integrals [Imamura-Sasamoto '11]. Equivalence to nested formula is non-trivial.